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STRONG NONLINEARITY REGIMES OF TWO-MODE PHOTOASSOCIATION OF ATOMIC BOSE-CONDENSATES

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Two distinct strongly non-linear scenarios of two-mode one-color photoassociation of atomic Bose-Einstein condensates corresponding to large and small detunings of the applied laser field's frequency are revealed. By examining arbitrary external field configurations, we show that the photoassociation process in the first case is almost non-oscillatory in time while in the second case the evolution of the system displays strongly pronounced Rabi-type oscillations.

In the present work we study the dynamics of molecule formation at coherent interaction of an atomic Bose-Einstein condensate with an optical laser field. We consider the basic case of one-color two-mode photoassociation described by the following system of coupled first-order nonlinear equations [1]:

$$i\frac{da_1}{dt} = Ue^{-i\delta}\bar{a}_1 a_2, \quad i\frac{da_2}{dt} = \frac{U}{2}e^{+i\delta}a_1 a_1, \tag{1}$$

which treats atomic and molecular condensates as classical fields. Here a_1 and a_2 are the free atomic and bound molecular states' amplitudes, respectively, $\bar{a}_1$ is the complex conjugate to a_1, $U(t)$ is the Rabi frequency, and $\delta(t)$ is the detuning modulation function defined as the integral of the frequency detuning. The initial conditions imposed are $|a_2(-\infty)|^2 = 0$, $|a_1(-\infty)|^2 = 1$. System (1) possesses a first integral indicating the conservation of the number of particles: $|a_1|^2 + 2|a_2|^2 = \text{const} = I_N$. Hereafter, for definiteness, we put $I_N = 1$.

In the present study we focus on the case of strong interaction when the nonlinearity is most expressed. Assuming arbitrary field configurations, we discuss possible general limit cases. It turns out that two distinct regimes are possible. In the first case, corresponding to the large frequency detunings and high field intensities, the transition of atoms into the bound molecular state takes place almost non-oscillatory in time (only weakly expressed oscillations between the two population modes are observed). On the contrary, in the small detuning regime the hybrid atomic-molecular system displays large amplitude Rabi-type oscillations of the populations. We illustrate the peculiarities of these two regimes using the Rosen-Zener non-crossing model [2]

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$$U(t)=\frac{U_0}{\mathrm{ch}(t/\tau)}\,,\ \delta_t(t)=2\delta_1 \tag{2}$$

and first Demkov-Kunike quasi-linear level-crossing model [3]

$$U(t)=\frac{U_0}{\mathrm{ch}(t/\tau)}\,,\ \delta_t(t)=2\delta_1\mathrm{th}(t/\tau) \tag{3}$$

(without loss of generality, we put $\tau=1$). Multiple level-crossing models are not considered.

It has been previously shown that the dynamics of the molecular state probability $p(t)=|a_2|^2$ is described by the following nonlinear ordinary differential equation of the third order [4]:

$$p_{ttt}-\left(\frac{\delta_{tt}}{\delta_t}+2\frac{U_t}{U}\right)p_{tt}+\left[\delta_t^2+4U^2(1-3p)-\left(\frac{U_t}{U}\right)_t+\frac{U_t}{U}\left(\frac{\delta_{tt}}{\delta_t}+\frac{U_t}{U}\right)\right]p_t+\frac{U^2}{2}\left(\frac{\delta_{tt}}{\delta_t}-\frac{U_t}{U}\right)(1-8p+12p^2)=0\,. \tag{4}$$

As is seen, the nonlinearity of the problem is determined by the current value of $p(t)$. Hence, the strong nonlinearity regime may be defined as a regime in which the probability $p(t)$ admits (relatively) large values during the evolution of the system (note that because of the normalization constraint p cannot exceed $1/2$).

Eq. (4) is rather complicated; however, it is considerably simplified in the case of constant field amplitude. Fortunately, system (1) for any model with varying field amplitude can be reduced to an equivalent system with constant field amplitude via the transformation of the independent variable [5]

$$z(t)=\int_{t_0}^{t}\frac{U(t')}{U_0}dt' \tag{5}$$

(usually we make the choice $U_0=\mathrm{Max}[U(t)]$). This transformation changes equation (4) to the following much simpler form:

$$p_{zzz}-\frac{\delta^*_{zz}}{\delta^*_z}p_{zz}+\left[\delta^{*2}_z+4U_0^2(1-3p)\right]p_z+\frac{U_0^2}{2}\frac{\delta^*_{zz}}{\delta^*_z}\left(1-8p+12p^2\right)=0\,, \tag{6}$$

where the effective detuning δ^*_z is defined as

$$\delta^*_z(z(t))=\delta_t(t)\frac{U_0}{U(t)}\,. \tag{7}$$

The nonlinear terms in Eq. (6) are proportional to the field intensity U_0^2. Hence, one may expect that the strong nonlinearity regime corresponds to high field strengths and we thus suppose that U_0^2 is a large parameter. Note further that the function δ^{*2}_z can also adopt large values (e.g., in the Landau-Zener model $\delta^{*2}_z\sim t^2$ [6]). Therefore, we suppose that the leading terms in Eq. (6) are the last two so that we neglect, for a while, the first two terms thus arriving at the following nonlinear equation of the first order:

$$\left[\delta_z^{*2}+4U_0^2(1-3p_0)\right]p_{0z}+\frac{U_0^2}{2}\frac{\delta_{zz}^*}{\delta_z^*}\left(1-8p_0+12{p_0}^2\right)=0\,. \tag{8}$$

This equation has trivial stationary solutions $p_0=1/2$ and $p_0=1/6$, which, as it will be shown below, play a prominent role in determination of the asymptotes of the solution to full Eq. (6).

In spite of its complexity, the limit Eq. (8) is exactly solved for an arbitrary function δ_z^*. Indeed, it is possible to find such transformation of the independent variable $z\to s$ that reduces nonlinear Eq. (8) to a linear equation for the new variable s if s is considered as a ***dependent*** variable, p_0 being now the independent variable. This is achieved by choosing $s={U_0}^2/\delta_z^{*2}$. After simple transformations we then arrive at the following equation:

$$\frac{{U_0}^2}{\delta_z^{*2}}=\frac{C+p_0(p_0-1/2)^2}{9(p_0-1/6)^2(p_0-1/2)^2}\,, \tag{9}$$

where C is a constant that should be defined from initial conditions. This equation leads to an important general conclusion that the molecular state probability is strictly equal (within the limitations of the limit equation (8)) to $1/6$ at the frequency resonance crossing point. It then follows that for non-crossing models the molecular state probability cannot exceed $1/6$, hence, the sweep through the resonance is a necessary condition to create a considerable molecular population.

A little examination shows that to define the constant C in Eq. (9) the behavior of the function δ_z^{*2} at $t\to-\infty$ should be considered. It is not difficult to verify that for the Rosen-Zener and first Demkov-Kunike models (2)-(3) holds $\lim\limits_{t\to-\infty}\left|\delta_z^*\right|=\infty$. Imposing now the initial condition $p_0(t=-\infty)=0$ we get $C=0$. But even now p_0 is not defined unambiguously. To make the right choice one should compare p_0 with the numerical solution to the exact equation (4). The result is that the following branch of the solution to Eq. (9) should be taken:

$$p_0=\frac{1}{6}+\frac{\delta_z^*}{18U_0}\left(\frac{\delta_z^*}{U_0}-S\sqrt{\left(\frac{\delta_z^*}{U_0}\right)^2+6}\right), \tag{10}$$

where

$$S=\operatorname{sgn}\left(\lim_{t\to-\infty}\delta_t(t)\right). \tag{11}$$

Note that if the limit solution (10) during its evolution exceeds the maximal allowed value $1/2$, it must be combined with the trivial solution $p_0=1/2$. This happens, e.g., for the first Demkov-Kunike model (see Fig. 1).

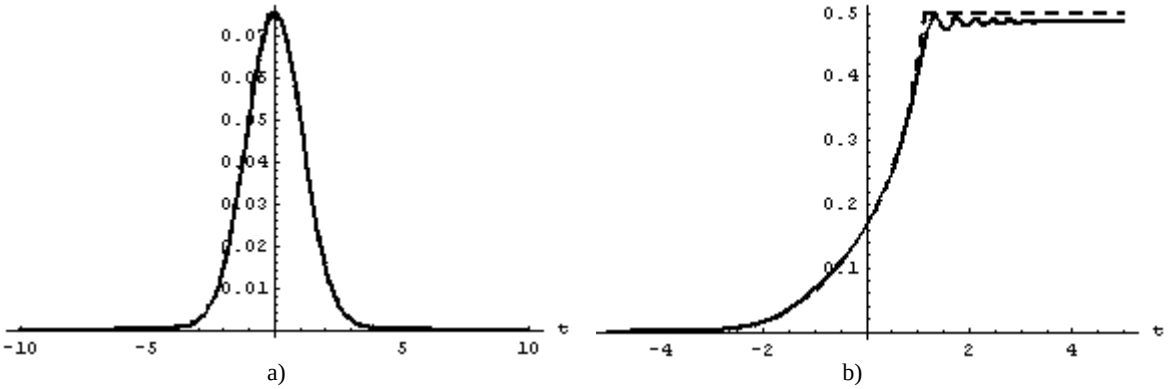


Fig. 1. Molecular state probability as a function of time. Solid line - numerical solution, dashed line -limit solution (10). a) Rosen-Zener model. $U_0 = 20$, $\delta_1 = 10$. b) The first Demkov-Kunike model. $U_0 = 20$, $\delta_1 = 10$.

The derived limit solution (10) is rather accurate approximation. This is shown in Fig. 1, where we compare this solution with the numerical solution to Eq. (4). Furthermore, this solution allows drawing several general qualitative conclusions of practical significance**.**

Suppose that the limit solution p_0 always remains less than $1/2$ or, equivalently, $\delta_t / U \le \sqrt{2}$, if $S = -1$, and $\delta_t / U \ge -\sqrt{2}$, if $S = 1$. In this case if

$$\lim_{t \to -\infty} \delta_z^*(z(t)) = \lim_{t \to +\infty} \delta_z^*(z(t)), \tag{12}$$

then after interaction the system will return to its initial pure atomic state. This happens, for instance, when the external field configuration is defined by the Rosen-Zener model (2). Now, let in addition δ_z^* remains restricted in the neighborhood of $t = +\infty$ for any finite values of the detuning and Rabi frequency parameters. Then the final transition probability tends to $1/6$, when $U_0 = \max[U(t)]$ tends to infinity. This result proves that the application of high field intensities is not always effective to achieve large final molecular population.

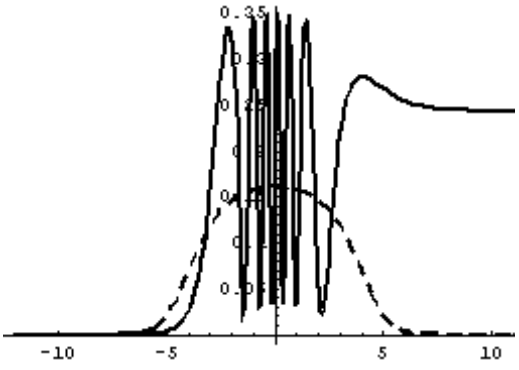


Fig. 2. Molecular state probability for the Rosen-Zener model at $U_0 = 10$, $\delta_1 = 0.2$. Solid line - numerical solution, dashed line - limit solution (10).

The common feature of the limit solutions for the Rosen-Zener and first Demkov-Kunike models is their non-oscillatory behavior. To find the conditions when such behavior is the case the approximate solutions are compared with the numerical ones. The analysis shows that the limit

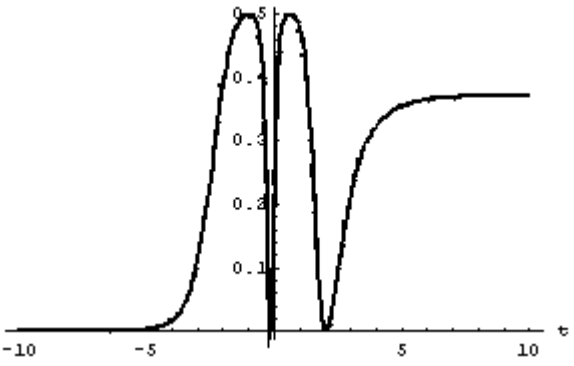


Fig. 3. Molecular state probability for the Rosen-Zener model. Solid line - numerical solution, dashed line - approximate solution (14). $U_0 = 7.2$, $\delta_1 = 0.004$, $\tilde{\delta}_0 \approx 0.0524$.

solution displays strong discrepancy as δ_1 and U_0 are lowered (Fig. 2). Finally, we conclude that solution (10) is a good approximation when $U_0 >> 1$ and $\delta_1 > 1$.

Thus, to construct an approximate solution for the parameter variation range $U_0 >> 1$ and $\delta_1 < 1$ a different approach should be developed. The numerical examination shows that in this case the behavior of the system is much more "unstable": the time evolution of the molecular state probability reveals oscillations with large amplitude and varying frequency. So that, this limit presents a more complicated problem. The valid approximation for this case is constructed by factorizing the exact third-order equation (6) for molecular state probability as follows:

$$\left(\frac{d}{dz} - \frac{\delta^*_{zz}}{\delta^*_z}\right)\left(p_{zz} - \frac{U_0^2}{2}\left(1 - 8p + 12p^2\right) + \delta^{*2}_z p\right) - \delta^*_z \delta^*_{zz} p = 0. \qquad (13)$$

In the case of small detuning the last term of this equation is rather small. Furthermore, it should be taken into account that the main change in the dynamics of the system takes place in the neighborhood of the resonance. Hence, the role of the last term in Eq. (13) is rather restricted and it can be neglected. Further, we integrate once the truncated equation, expand its coefficients in power series in the vicinity of the point $z = 0$ and keep only the zeroth order terms. As a result, we arrive at the following approximate solution, in terms of the Jacobi elliptic sine [7]:

$$\tilde{p} = p_1 \,\mathrm{sn}^2[\sqrt{p_2} U_0 (z - z_0); m], \qquad (14)$$

where

$$p_{1,2} = \frac{1}{2}\left(\frac{\delta^{*2}_0}{4U_0^2} + 1\right) \mp \sqrt{\frac{1}{4}\left(\frac{\delta^{*2}_0}{4U_0^2} + 1\right)^2 - \frac{1}{4}}, \quad \delta^*_0 = \lim_{z \to 0} \delta^*_z \quad \text{and} \quad m = \frac{p_1}{p_2}. \qquad (15)$$

The function (14) is periodic with the period given as

$$T(m) = \frac{\pi}{\sqrt{p_2} U_0} \cdot {}_2F_1(1/2, 1/2; 1; m). \qquad (16)$$

The comparison of the obtained solution with the numerical one shows that the coincidence is good enough only at the beginning of the interaction. However, the result can be essentially improved by adjusting the parameter δ^*_0 involved in the solution (14)-(15). First, we numerically show that there can always be found such parameter value $\delta^*_0 = \tilde{\delta}_0$ for which the approximate solution almost perfectly coincides with the numerical one (Fig. 3). Further, by

application of the Linstedt-Poincare method [8], an analytic expression for $\tilde{\delta}_0$ as a function of the detuning and Rabi frequency parameters can be found (in particular, for the Rosen-Zener model we get $\tilde{\delta}_0 = 1.2U_0\delta_1$).

The solution (14) is analogous to the nonlinear Rabi-solution [9]. As it is immediately seen, it is of universal form for arbitrary pulse shape and detuning modulation functions; the change of the laser field's configuration only affects the argument and the expression for $\tilde{\delta}_0$ leaving the function itself unchanged. Hence, the qualitative behavior of the system in this regime is less sensitive to the concrete form of the laser excitation. Another interesting feature is a subtle dependence of the oscillation frequency of the molecular state probability on the parameters of the modulation of the laser field [see Eq. (16)]. In particular, the dependence of the frequency on the field amplitude is nonlinear, and moreover, oscillatory.

Thus, we have analyzed the strong interaction limit ($U_0 >> 1$) of the two-mode one-color photoassociation of an atomic Bose-Einstein condensate and developed a general strategy for solving the problem for arbitrary pulse shape and detuning modulation functions. We have shown that there are two distinct scenarios of the dynamics of the system - the large and small detuning regimes. The main peculiarity of the photoassociation process in the large detuning regime is its almost non-oscillatory behavior, i.e., only weak oscillations between the atomic and molecular populations occur. This regime is satisfactory approximated by the non-oscillatory limit solution (10). On the contrary, in the small detuning regime the evolution of the system is essentially oscillatory; in this case the approximate solution is expressed in terms of the oscillatory Jacobi-sine function [Eq. (14)]. The origin of the oscillations can be qualitatively understood by examining the effective interaction time for the models under consideration (2)-(3). If the detuning parameter δ_1 is large, then in the regions far from the point $t = 0$, where the field amplitude is approximately zero, the interaction is rather weak, and the system does not change its state considerably. However, in the case of small δ_1 the effective interaction time, which is proportional to $1/\delta_1$, is large, hence, during this time period the system will considerably change its state despite the smallness of the Rabi frequency: large-amplitude Rabi-type oscillations will start.

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